An Identity Involving Partitional Generalized Binomial Coefficients

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Lemma 1.

Let z and t be arbitrary scalars. Then

$$\mathrm{etr}(-ZT)\,{}_0F_0^{(p)}(Z,\ T) = \mathrm{etr}(-(Z + zI_p)(T + tI_p))\,{}_0F_0^{(p)}(Z + zI_p,\ T + tI_p)\ .$$

Proof:

Consider the exponent in the integrand of (2.1):

$$\begin{aligned}\mathrm{tr}\ ZH'TH &= \mathrm{tr}(Z + zI_p)H'(T + tI_p)H - z\mathrm{tr}(T) - t\ \mathrm{tr}(T) - pzt \\ &= \mathrm{tr}(Z + zI_p)H'(T + tI_p)H + \mathrm{tr}(ZT) - \mathrm{tr}(Z + zI_p)(T + tI_p).\end{aligned}$$

The result now is immediate. □

The main result is the following.

Theorem 1.

Let the generalized binomial coefficients be defined by (1.1), and let $\varkappa \in \mathcal{P}_k$. Then

$$\sum_{\ell=k}^{\infty} \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\varkappa} C_\lambda(Z)/\ell! = C_\varkappa(Z)\ \mathrm{etr}(Z)/k!\ . \tag{2.4}$$

Proof:

By (2.3) a generating function for $C_\varkappa(Z)\ \mathrm{etr}(Z)$ can be taken to be

$$\mathrm{etr}(Z)\,{}_0F_0^{(p)}(Z,\ T) = \sum_{k=0}^{\infty} (1/k!) \sum_{\varkappa\in\mathcal{P}_k} [C_\varkappa(Z)\ \mathrm{etr}(Z)]C_\varkappa(T)/C_\varkappa(I_p).$$

Using Lemma 1 with $z = 0$ and $t = 1$ we also have

$$\begin{aligned}\mathrm{etr}(Z)\,{}_0F_0^{(p)}(Z,\ T) &= {}_0F_0^{(p)}(Z,\ I_p + T) \\ &= \sum_{\ell=0}^{\infty} (1/\ell!) \sum_{\lambda\in\mathcal{P}_\ell} C_\lambda(Z)C_\lambda(I_p + T)/C_\lambda(I_p).\end{aligned}$$

From (1.1) we thus have

$$\begin{aligned}\mathrm{etr}(Z)\,{}_0F_0^{(p)}(Z,\ T) &= \sum_{\ell=0}^{\infty} (1/\ell!) \sum_{\lambda\in\mathcal{P}_\ell} C_\lambda(Z) \sum_{k=0}^{\ell} \sum_{\varkappa\in\mathcal{P}_k} \binom{\lambda}{\varkappa} C_\varkappa(T)/C_\varkappa(I_p) \\ &= \sum_{k=0}^{\infty} \sum_{\varkappa\in\mathcal{P}_k} \Big[\sum_{\ell=k}^{\infty} \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\varkappa} C_\lambda(Z)/\ell!\Big] C_\varkappa(T)/C_\varkappa(I_p).\end{aligned}$$

Comparing coefficints of $C_\varkappa(T)/C_\varkappa(I_p)$ the theorem is proved. □

1. Summary.

Constantine (1966) defined a generalized "binomial" type coefficient $\binom{\lambda}{\kappa}$ (his notation was $a_{\lambda,\kappa}$) by the expansion

$$(1.1) \qquad C_\lambda(I_p + Z)/C_\lambda(I_p) = \sum_{k=0}^{\ell} \sum_{\kappa \in \mathcal{P}_k} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p)$$

where $\mathcal{P}_k$ is the set of all partitions $\kappa = (k_1, k_2, \ldots, k_p)$, $k_{i-1} \geq k_i$, $\Sigma_{i=1}^p k_i = k$, $\lambda \in \mathcal{P}_\ell$, and the $C_\kappa(Z)$'s are zonal polynomials as defined by James (1964). The principal result derived below is the identity

$$(1.2) \qquad \sum_{\ell=k}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} \binom{\lambda}{\kappa} C_\lambda(Z)/\ell! = C_\kappa(Z)\ \mathrm{etr}(Z)/k!$$

where $\mathrm{etr}(Z) = \exp(\mathrm{tr}\ Z)$. This identity is extended to analogous sums involving generalized Laguerre polynomials, generalized Hermite polynomials, and other polynomials. In addition it is used to derive several recurrence relations among the $\binom{\lambda}{\kappa}$'s.

2. The Principal Result.

James (1964) defined the function

$$(2.1) \qquad {}_0F_0^{(p)}(Z, T) = V_p^{-1} \int_{O(p)} \mathrm{etr}(ZH'TH)\ (dH)$$

where $O(p)$ is the group of orthogonal p by p matrices with invariant (Haar) measure (dH) with total content V_p. Using the identity

$$(2.2) \qquad V_p^{-1} \int_{O(p)} C_\kappa(ZH'TH)\ (dH) = C_\kappa(Z)C_\kappa(T)/C_\kappa(I_p)$$

he derives the expansion

$$(2.3) \qquad {}_0F_0^{(p)}(Z, T) = \sum_{k=0}^{\infty} (1/k!) \sum_{\kappa \in \mathcal{P}_k} C_\kappa(Z)C_\kappa(T)/C_\kappa(I_p).$$

Thus ${}_0F_0^{(p)}(Z, T)$ can be considered to be a generating function for the zonal polynomials $C_\kappa(Z)$.

Several expansions in zonal polynomials of $P(Z)\ \mathrm{etr}(Z)$, where $P(Z)$ is a symmetric polynomial in the latent roots of Z, have appeared in the literature (Suguira and Fujikoshi (1969), Fujikoshi (1970)). Since any such $P(Z)$ can be expressed in terms of zonal polynomials, the coefficients appearing in these expansions can be expressed in terms of generalized binomial coefficients. Using, for example, the summary in Fujikoshi (1970), we can thus find all $\binom{\lambda}{\kappa}$ for $\kappa \in \mathcal{P}_k$, $k \leq 3$. It is convenient to use a notation that differs from that which has generally been used previously.

Definition.

Let $\kappa = (k_1, k_2, \ldots, k_p) \in \mathcal{P}_k$. Then

$$d_0(\kappa) \equiv 1,\ d_r(\kappa) \equiv \sum_{i=1}^{p} \sum_{j=1}^{k_i} (j - \tfrac{1}{2}(i+1))^{r-1},\ r = 1,2,\ldots. \tag{2.5}$$

Clearly $d_1(\kappa) = k$. Moreover, it is easily checked that Suguira and Fujikoshi's (1969) $a_1(\kappa)$ and $a_2(\kappa)$ and Fujikoshi's (1970) $a_3(\kappa)$ can be expressed as

$$\begin{aligned} a_1(\kappa) &= 2d_2(\kappa) \\ a_2(\kappa) &= 12d_3(\kappa) + k \\ a_3(\kappa) &= 8d_4(\kappa) + 2d_2(\kappa) \quad . \end{aligned} \tag{2.6}$$

Conversely,

$$\begin{aligned} d_1(\kappa) &= k \\ d_2(\kappa) &= \tfrac{1}{2}a_1(\kappa) \\ d_3(\kappa) &= (1/12)(a_2(\kappa) - k) \\ d_4(\kappa) &= (1/8)(a_3(\kappa) - a_1(\kappa)) \ . \end{aligned} \tag{2.7}$$

Using (2.6),the series of Suguira and Fujikoshi (1969) yield the results $[s_r \equiv \mathrm{tr}(Z^r)]$

$$s_1^k\ \mathrm{etr}(Z)/k! = \sum_{\ell=k}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} \binom{\ell}{\ell-k} C_\lambda(Z)/\ell! \tag{2.8}$$

$$(2.9) \qquad s_2 \ \mathrm{etr}(Z)/2! = \sum_{\ell=k}^{\infty} \sum_{\lambda\in P_\ell} d_2(\lambda) C_\lambda(Z)/\ell!$$

$$(2.10) \qquad s_1 s_2 \ \mathrm{etr}(Z)/3! = \sum_{\ell=k}^{\infty} \sum_{\lambda\in P_\ell} (1/3)(\ell-2) d_2(\lambda) C_\lambda(Z)/\ell!$$

$$(2.11) \qquad s_3 \ \mathrm{etr}(Z)/3! = \sum_{\ell=k}^{\infty} \sum_{\lambda\in P_\ell} [\tfrac{1}{2} d_3(\lambda) - \tfrac{1}{4} d_2(\lambda) - \tfrac{1}{4}\tbinom{\ell}{2}] C_\lambda(Z)/\ell! \ .$$

Using the explicit expressions for low order $C_\varkappa(Z)$ in James (1964), (2.8) through (2.11), together with (2.4), yield the following explicit expressions for $\binom{\lambda}{\varkappa}$, $\varkappa \in P_k$, $k \leq 3$ and arbitrary $\lambda \in P_\ell$.

$$\begin{aligned}
&\tbinom{\lambda}{(0)} = 1\\
&\tbinom{\lambda}{(1)} = \ell\\
&\tbinom{\lambda}{(2)} = (\tfrac{1}{3})[\tbinom{\ell}{2} + 2d_2(\lambda)]\\
(2.12) \qquad &\tbinom{\lambda}{(1^2)} = (\tfrac{1}{3})[2\tbinom{\ell}{2} - 2d_2(\lambda)]\\
&\tbinom{\lambda}{(3)} = (\tfrac{1}{15})[\tbinom{\ell}{3} + 2(\ell-2)d_2(\lambda) + 4d_3(\lambda) - 2(\tbinom{\ell}{2} + d_2(\lambda))]\\
&\tbinom{\lambda}{(21)} = (\tfrac{1}{15})[9\tbinom{\ell}{3} + 3(\ell-2)d_2(\lambda) - 9d_3(\lambda) + (\tfrac{9}{2})(\tbinom{\ell}{2} + d_2(\lambda))]\\
&\tbinom{\lambda}{(1^3)} = (\tfrac{1}{15})[5\tbinom{\ell}{3} - 5(\ell-2)d_2(\lambda) + 5d_3(\lambda) - (\tfrac{5}{2})(\tbinom{\ell}{2} + d_2(\lambda))] \ .
\end{aligned}$$

Methods for deriving these directly, using recurrence relations among the $\binom{\lambda}{\varkappa}$'s will be given in Section 4.

3. <u>Other Analogous Expansions</u>.

Theorem 1 is the basis of several analogous identities involving other polynomials that can be defined by means of zonal polynomials.

<u>Theorem 2</u>.

Let Z be such that $\|Z\| < 1$, $\varkappa \in P_k$ and let b be arbitrary. Then

$$(3.1) \qquad \sum_{\ell=k}^{\infty} \sum_{\lambda\in P_\ell} \tbinom{\lambda}{\varkappa} (b)_\lambda C_\lambda(Z)/\ell! = [\det(I_p - Z)]^{-b} (b)_\varkappa C_\varkappa((I_p - Z)^{-1}Z)/k!$$

where $(b)_\rho = \prod_{i=1}^{p} (b - \frac{1}{2}(i-1))_{r_i}$, $\rho = (r_1, \ldots, r_p) \in \mathcal{P}_r$, and $(x)_r = x(x+1) \ldots (x+r-1)$.

Proof:

Consider the generating function for the left hand side of (3.1):

$$\sum_{k=0}^{\infty} \sum_{\varkappa \in \mathcal{P}_k} [\sum_{\ell=k}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} \binom{\lambda}{\varkappa} (b)_\lambda C_\lambda(Z)/\ell!] C_\varkappa(T)/C_\varkappa(I_p)$$

$$= \sum_{\ell=0}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} [(b)_\lambda C_\lambda(Z)/\ell!] \sum_{k=0}^{\ell} \sum_{\varkappa \in \mathcal{P}_k} \binom{\lambda}{\varkappa} C_\varkappa(T)/C_\varkappa(I_p)$$

$$= \sum_{\ell=0}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} (b)_\lambda C_\lambda(Z) C_\lambda(I_p + T)/[C_\lambda(I_p)\ell!], \text{ by } (1.1),$$

$$= \sum_{\ell=0}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} [(b)_\lambda/\ell!] V_p^{-1} \int_{O(p)} C_\lambda(ZH'(I_p + T)H)(dH), \text{ by } (2.2).$$

Reversing the order of integration and summation and observing that $[\det(I-S)]^{-b} = \Sigma_{\ell=0}^{\infty} \Sigma_{\lambda \in \mathcal{P}_\ell} (1/\ell!)(b)_\lambda C_\lambda(S)$ by Constantine (1963), we obtain for the generating function

$$V_p^{-1} \int_{O(p)} [\det(I_p - ZH'(I_p + T)H)]^{-b}(dH)$$

$$= [\det(I_p - Z)]^{-b} V_p^{-1} \int_{O(p)} [\det(I_p - (I_p - Z)^{-1} ZH'TH)]^{-b}(dH)$$

$$= [\det(I_p - Z)]^{-b} \sum_{k=0}^{\infty} \sum_{\varkappa \in \mathcal{P}_k} [(b)_\varkappa/k!] V_p^{-1} \int_{O(p)} C_\varkappa((I_p - Z)^{-1} ZH'TH)(dH)$$

$$= [\det(I_p - Z)]^{-b} \sum_{k=0}^{\infty} \sum_{\varkappa \in \mathcal{P}_k} [(b)_\varkappa C_\varkappa((I_p - Z)^{-1}Z)/k!] C_\varkappa(T)/C_\varkappa(I_p).$$

Comparing coefficient of $C_\varkappa(T)/C_\varkappa(I_p)$ we have the result. □

Corollary.

Let f_λ and $g_\varkappa$ be such that

$$(3.2) \qquad \sum_{\ell=0}^{\infty} \sum_{\lambda \in \mathcal{P}_\ell} f_\lambda C_\lambda(Z)/\ell! = [\sum_{k=0}^{\infty} \sum_{\varkappa \in \mathcal{P}_k} g_\varkappa C_\varkappa(Z)/k!] \operatorname{etr}(Z).$$

Then

$$(3.3)\qquad \sum_{\ell=0}^{\infty}\sum_{\lambda\in P_\ell}(b)_\lambda f_\lambda C_\lambda(Z)/\ell! = [\det(I-Z)]^{-b}\sum_{k=0}^{\infty}\sum_{\varkappa\in P_k}(b)_\varkappa g_\varkappa C_\varkappa((I-Z)^{-1}Z)/k!\ .$$

Proof:

(3.2) and Theorem 1 imply the identity

$$f_\lambda = \sum_{k=0}^{\ell}\sum_{\varkappa\in P_k} g_\varkappa \binom{\lambda}{\varkappa},\ \lambda \in P_\ell\ .$$

When this is substituted in the left hand side of (3.3), then (3.1) gives the right hand side of (3.3). □

By the Corollary, Fujikoshi's (1970) Equations 2.6 through 2.10 imply his Equations 2.16 through 2.20 when one expresses the $C_\varkappa((I-Z)^{-1}Z)$ in terms of monomials in $\mathrm{tr}[(I_p-Z)^{-1}Z]^r$.

Hayakawa (1969) defines a family of polynomials $P_\varkappa(T, A)$ in a p by n matrix T and an n by n symmetric matrix A by

$$(3.4)\qquad \mathrm{etr}(-TT')P_\varkappa(T, A) = (-1)^k \pi^{-\frac{1}{2}pn}\int_U \mathrm{etr}(-2i\ TU')\mathrm{etr}(-UU')C_\varkappa(UAU')dU,$$

where $U = [u_{ij}]$ is a real p by n matrix, and the domain of integration is over $-\infty < u_{ij} < +\infty$, all u_{ij}; $dU = \prod_{i=1}^{p}\prod_{j=1}^{n} du_{ij}$. Particular cases of $P_\varkappa(T, A)$ are (Hayakawa (1969), Theorem 6, corrected)

$$(3.5)\qquad P_\varkappa(0, A) = (-1)^k(p/2)_\varkappa C_\varkappa(A)$$

$$(3.6)\qquad P_\varkappa(T, I_n) = H_\varkappa(T),$$

where $H_\varkappa(T)$ is a particular form of the generalized Hermitian polynomial defined by Herz (1955) and Hayakawa (1969).

Theorem 3.

Let $\varkappa \in P_k$, $\|A\| < 1$, and let $(I_n - A)^{\frac{1}{2}}$ be a symmetric square root of $I_n - A$. Then

$$(3.7)\qquad \sum_{\ell=k}^{\infty}\sum_{\lambda\in\mathcal{P}_\ell}(-1)^{\ell}\binom{\lambda}{\varkappa}P_\lambda(T, A)/\ell!$$

$$= (-1)^k \operatorname{etr}(-(I_n - A)^{-1}ATT')[\det(I_n - A)]^{-\frac{1}{2}p}$$

$$\cdot P_\varkappa(T(I_n - A)^{-\frac{1}{2}}, (I_n - A)^{-\frac{1}{2}}A(I_n - A)^{-\frac{1}{2}})/k! \ .$$

Proof:

From (3.4) the left hand side of (3.7) is

$$\operatorname{etr}(TT')\pi^{-\frac{1}{2}pn}\int_U \{\operatorname{etr}(-2i\ TU')\operatorname{etr}(-UU')\sum_{\ell=k}^{\infty}\sum_{\lambda\in\mathcal{P}_\ell}\binom{\lambda}{\varkappa}C_\lambda(UAU')/\ell!\}dU.$$

By Theorem 1, this is

$$\operatorname{etr}(TT')\pi^{-\frac{1}{2}pn}\int_U \{\operatorname{etr}(-2i\ TU')\operatorname{etr}(-UU')\operatorname{etr}(UAU')C_\varkappa(UAU')\}dU/k!$$

$$= \operatorname{etr}(TT')\pi^{-\frac{1}{2}pn}\int_U \{\operatorname{etr}(-2i\ TU')\operatorname{etr}(-U(I_n - A)U')C_\varkappa(UAU')\}dU/k! \ .$$

Changing variables to $V = U(I_n - A)^{\frac{1}{2}}$, this becomes (since $dU = [\det(I_n - A)]^{-\frac{1}{2}p}dV$)

$$[\det(I_n - A)]^{-\frac{1}{2}p}\operatorname{etr}(TT')\pi^{-\frac{1}{2}pn}\int_V \{\operatorname{etr}(-2i\ T(I_n - A)^{-\frac{1}{2}}V')$$

$$\cdot \operatorname{etr}(-VV')C_\varkappa(V(I_n - A)^{-\frac{1}{2}}A(I_n - A)^{-\frac{1}{2}}V')\}dV/k!$$

$$= [\det(I_n - A)]^{-\frac{1}{2}p}\operatorname{etr}(TT' - T(I_n - A)^{-1}T')P_\varkappa(T(I_n - A)^{-\frac{1}{2}},$$

$$(I_n - A)^{-\frac{1}{2}}A(I_n - A)^{-\frac{1}{2}})/k!$$

by (3.4). Since $\operatorname{tr}(TT' - T(I_n - A)^{-1}T') = \operatorname{tr}((I_n - (I_n - A)^{-1})TT') = -\operatorname{tr}((I_n - A)^{-1}ATT')$, the result follows. □

Corollary.

Let f_λ and $g_\varkappa$ satisfy (3.2). Then

$$(3.8)\qquad \sum_{\ell=0}^{\infty}\sum_{\lambda\in\mathcal{P}_\ell}(-1)^{\ell}f_\lambda P_\lambda(T, A)/\ell! = [\det(I_n - A)]^{-\frac{1}{2}p}\operatorname{etr}(-(I_n - A)^{-1}ATT')$$

$$\cdot \sum_{k=0}^{\infty}\sum_{\varkappa\in\mathcal{P}_k}(-1)^k g_\varkappa P_\varkappa(T(I_n - A)^{-\frac{1}{2}}, (I_n - A)^{-\frac{1}{2}}A(I_n - A)^{-\frac{1}{2}})/k! \ .$$

Proof:

This is proved the same way as the Corollary to Theorem 2. □

Note that from (3.5), Theorem 3 implies Theorem 2 for the case $b = \frac{1}{2}p$.

Theorem 4.

Let $|x| < 1$, $\kappa \in \mathcal{P}_k$, and let T be a p by n matrix. Then

$$(3.9)\qquad \sum_{\ell=k}^{\infty} \sum_{\lambda\in\mathcal{P}_\ell} (-1)^\ell \binom{\lambda}{\kappa} x^\ell H_\lambda(T)/\ell! = (-1)^k \operatorname{etr}(-x(1-x)^{-1}TT')(1-x)^{-\frac{1}{2}pn} \cdot [x(1-x)^{-1}]^k H_\kappa((1-x)^{-\frac{1}{2}}T)/k!\ .$$

Proof:

This follows directly from Theorem 3, observing that (3.4) and (3.6) imply that $x^\ell H_\lambda(T) = P_\lambda(T, xI_n)$. □

Corollary.

Let f_λ and g_κ satisfy (3.2). Then

$$(3.10)\qquad \sum_{\ell=0} \sum_{\lambda\in\mathcal{P}_\ell} (-1)^\ell f_\lambda x^\ell H_\lambda(T)/\ell! = \operatorname{etr}(-x(1-x)^{-1}TT')(1-x)^{-\frac{1}{2}pn} \sum_{k=0}^{\infty} \sum_{\kappa\in\mathcal{P}_k} g_\kappa (-1)^k \cdot [x(1-x)^{-1}]^k H_\kappa((1-x)^{-\frac{1}{2}}T)/k!\ .$$

Proof:

This is proved the same way as the Corollary to Theorem 2. □

Constantine (1966), following Herz (1955), defines generalized Laguerre polynomials to be, for $\lambda \in \mathcal{P}_\ell$ and $q = 2\gamma + p + 1$,

$$(3.11)\qquad L_\lambda^\gamma(Z) = \operatorname{etr}(Z)[\Gamma_p(\tfrac{1}{2}q)]^{-1} \int_{R>0} \operatorname{etr}(-R)(\det R)^\gamma \cdot {}_0F_1(\tfrac{1}{2}q;-RZ)C_\lambda(R)dR,$$

where the integral is over all p by p positive definite symmetric R and

$${}_0F_1(b;\ Z) = \sum_{k=0}^{\infty} \sum_{\kappa\in\mathcal{P}_k} 1/(k!(b)_\kappa)C_\kappa(Z)\ .$$

Hayakawa (1969) shows that $H_\kappa(T) = (-1)^k L_\kappa^{\frac{1}{2}n-p}(TT')$.

Theorem 5.

Let $|x| < 1$, $\kappa \in \mathcal{P}_k$, and let Z be a p by p symmetric matrix. Then

$$(3.12)\qquad \sum_{\ell=k}^{\infty}\sum_{\lambda\in\mathcal{P}_\ell}\binom{\lambda}{\kappa}x^\ell L_\lambda^\gamma(Z)/\ell! = (1-x)^{-\frac{1}{2}pq}\operatorname{etr}(-x(1-x)^{-1}Z)[x(1-x)^{-1}]^k L_\kappa^\gamma((1-x)^{-1}Z)/k!\ .$$

Proof:

By (3.11) and Theorem 1, the left hand side of (3.12) is

$$\operatorname{etr}(Z)[\Gamma_p(\tfrac{1}{2}q)]^{-1}\int_{R>0}\{\operatorname{etr}(-R)(\det R)^\gamma {}_0F_1(\tfrac{1}{2}q;\ -RZ)\operatorname{etr}(xR)C_\kappa(xR)\}dR/k!$$

$$= \operatorname{etr}(Z)[\Gamma_p(\tfrac{1}{2}q)]^{-1}\int_{R>0}\{\operatorname{etr}(-(1-x)R)(\det R)^\gamma {}_0F_1(\tfrac{1}{2}q;\ -RZ)C_\kappa(xR)\}dR/k!\ .$$

Changing variables to $S = (1-x)R$ and observing that $dR = (1-x)^{-\frac{1}{2}p(p+1)}dS$, this is equal to

$$(1-x)^{-p\gamma-\frac{1}{2}p(p+1)}[x(1-x)^{-1}]^k\operatorname{etr}(Z)[\Gamma_p(\tfrac{1}{2}q)]^{-1}\int_{S>0}\{\operatorname{etr}(-S)$$

$$\cdot(\det S)^\gamma {}_0F_1(\tfrac{1}{2}q;\ -S[(1-x)^{-1}Z])C_\kappa(S)\}dS/k!$$

$$= (1-x)^{-\frac{1}{2}pq}\operatorname{etr}(Z-(1-x)^{-1}Z)[x(1-x)^{-1}]^k L_\kappa^\gamma((1-x)^{-1}Z)/k!\ .$$

Since $1 - (1-x)^{-1} = -x(1-x)^{-1}$, (3.12) follows. □

Corollary.

Let f_λ and g_κ satisfy (3.2). Then

$$(3.13)\qquad \sum_{\ell=0}^{\infty}\sum_{\lambda\in\mathcal{P}_\ell} f_\lambda x^\ell L_\lambda^\gamma(Z)/\ell! = (1-x)^{-\frac{1}{2}pq}\operatorname{etr}(-x(1-x)^{-1}Z)\sum_{k=0}^{\infty}\sum_{\kappa\in\mathcal{P}_k} g_\kappa[x(1-x)^{-1}]^k L_\kappa^\gamma((1-x)^{-1}Z)/k!\ .$$

Proof:

This is proved the same way as the Corollary to Theorem 2. □

This Corollary, together with Fujikoshi's (1970) Equations 2.6 through 2.10, could be used to derive (and express more compactly) his Equations 4.1 through 4.7.

4. Recurrence Relations Among the $\binom{\lambda}{\varkappa}$.

Although no general formula for $\binom{\lambda}{\varkappa}$ is known at this time, various recurrence relations can be derived that enable one to determine particular cases, including all those in (2.8). The simplest is derivable almost directly from Theorem 1.

Lemma 2.

Let $0 \leq k \leq \ell$ and $\varkappa \in \mathcal{P}_k$. Then

$$\binom{\ell}{\ell-k} s_1^{\ell-k} C_\varkappa(Z) = \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\varkappa} C_\lambda(Z), \tag{4.1}$$

where $s_1 = \operatorname{tr} Z$.

Proof:

This follows from Theorem 1 by matching terms of equal degree on both sides of (2.4). □

Given expressions for $C_\varkappa(Z)$ and all $C_\lambda(Z)$, $\lambda \in \mathcal{P}_\ell$, in terms of monomials in $s_r = \operatorname{tr}(Z^r)$, (4.1) provides a simple means of computing $\binom{\lambda}{\varkappa}$ for small values of k and ℓ. Its application is simplified by the following.

Lemma 3.

Let $\lambda = (\ell_1,\ldots, \ell_p) \in \mathcal{P}_\ell$, $\varkappa = (k_1,\ldots, k_p) \in \mathcal{P}_k$. Then

$$\binom{\lambda}{\varkappa} = 0 \quad \text{if any } \ell_i < k_i,\ i = 1,\ldots, p. \tag{4.2}$$

Proof:

By (1.1)

$$C_\lambda(tI_p + Z)/C_\lambda(I_p) = \sum_{r=0}^{\ell} t^r \sum_{\varkappa\in\mathcal{P}_{\ell-r}} \binom{\lambda}{\varkappa} C_\varkappa(Z)/C_\varkappa(I_p).$$

But also

$$C_\lambda(tI_p + Z) = \sum_{r=0}^{\ell} (t^r/r!)\left[(d/dt)^r C_\lambda(Z + tI_p)\Big|_{t=0}\right].$$

Without loss of generality we can assume that $Z = \mathrm{diag}[z_1,\ldots, z_p]$. Then, for any function $f(Z)$, $(d/dt)f(Z + tI_p) = \epsilon f(Z + tI_p)$, where $\epsilon = \Sigma_{j=1}^{p}(\partial/\partial z_j)$, as in Muirhead (1970). Thus we have the identity

$$\sum_{\kappa\in\mathcal{P}_{\ell-r}} \binom{\lambda}{\kappa} C_\kappa(Z)/C_\kappa(I_p) = (1/r!)\epsilon^r C_\lambda(Z)/C_\lambda(I_p).$$

But by James and Constantine (197), as referenced by Muirhead (1970),

$$\epsilon C_\lambda(Z)/C_\lambda(I_p) = \sum_{i=1}^{p} \binom{\lambda}{\lambda_{(i)}} C_{\lambda_{(i)}}(Z)/C_{\lambda_{(i)}}(I_p),$$

where $\lambda^{(i)} = (\ell_1,\ldots, \ell_i-1, \ell_{i+1},\ldots, \ell_p)$, any $\lambda^{(i)}$ with $\ell_i - 1 < \ell_{i+1}$ being omitted from the sum. Thus the application of ϵ^r to $C_\lambda(Z)$ produces only $C_\kappa(Z)$, $\kappa = (k_1,\ldots, k_p) \in \mathcal{P}_k$ satisfying $k_i \leq \ell_i$, $i = 1,\ldots, p$. □

We note that (4.2) can be verified empirically up to $\ell = 8$ in the Table of Pillai and Jouris (1969).

A recurrence relation connecting $\binom{\lambda}{\kappa}$ and $\binom{\rho}{\kappa}$, $\lambda \in \mathcal{P}_\ell$, $\rho \in \mathcal{P}_r$, $\kappa \in \mathcal{P}_k$, $k \leq \ell \leq r$ is the following.

<u>Theorem 6.</u>

Let $\kappa \in \mathcal{P}_k$, $\rho \in \mathcal{P}_r$, $k \leq \ell \leq r$. Then

$$\sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\kappa}\binom{\rho}{\lambda} = \binom{r-k}{\ell-k}\binom{\rho}{\kappa} . \tag{4.3}$$

<u>Proof:</u>

By (4.1)

$$\begin{aligned}\binom{r}{r-k}^{-1} \sum_{\rho\in\mathcal{P}_r} \binom{\rho}{\kappa} C_\rho(Z) &= s_1^{r-k} C_\kappa(Z) = s_1^{r-\ell} s_1^{\ell-k} C_\kappa(Z)\\ &= \binom{\ell}{\ell-k}^{-1} \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\kappa} s_1^{r-\ell} C_\lambda(Z)\\ &= \binom{\ell}{\ell-k}^{-1}\binom{r}{r-\ell}^{-1} \sum_{\rho\in\mathcal{P}_r} \sum_{\lambda\in\mathcal{P}_\ell} \binom{\lambda}{\kappa}\binom{\rho}{\lambda} C_\rho(Z) .\end{aligned}$$

Matching coefficients of $C_\rho(Z)$, and observing that $\binom{\ell}{\ell-k}\binom{r}{r-\ell}/\binom{r}{r-k} = \binom{r-k}{\ell-k}$, gives the result. □

Taking $\kappa = (0)$ in (4.3), we obtain

$$(4.4) \qquad \sum_{\lambda\in\mathcal{P}_\ell} \binom{\rho}{\lambda} = \binom{r}{\ell}, \ \rho \in \mathcal{P}_r .$$

All the information concerning $\binom{\rho}{\lambda}$ that is provided by (4.3) is contained in the identities for $\kappa \in \mathcal{P}_{\ell-1}$. Multiplying both sides of (4.3) by $\binom{\kappa}{\tau}$ for $\tau \in \mathcal{P}_t$, $t < k$, and summing over all $\kappa \in \mathcal{P}_k$ results in (4.3) for τ. Thus the identities for $k < \ell - 1$ depend linearly on those for $k = \ell - 1$, and hence (4.3) establishes at most $n_{\ell-1}$ relations between the $\binom{\rho}{\lambda}$, where n_k is the number of $\kappa \in \mathcal{P}_k$. Thus for $\ell > 1$, (4.3) is insufficient to generate all $\binom{\rho}{\ell}$.

For $\ell = 3$, using the Table in the Appendix to Constantine (1966), (4.3) reduces to $3\binom{\rho}{(3)} + (4/3)\binom{\rho}{(21)} = (r-2)\binom{\rho}{(2)}$ and $(5/3)\binom{\rho}{(21)} + 3\binom{\rho}{(1^3)}$ $= (r-2)\binom{\rho}{(1^2)}$. It can be verified that the expressions in (2.12) satisfy these relations.

Another set of recurrence relations can be derived as follows.

Lemma 4.

Let $\lambda \in \mathcal{P}_\ell$ and let a and b be arbitrary. Then

$$(4.5) \qquad \sum_{k=0}^{\ell} (-1)^k \sum_{\kappa\in\mathcal{P}_k} \binom{\lambda}{\kappa}(a)_\kappa/(b)_\kappa = (b-a)_\lambda/(b)_\lambda .$$

Proof:

Kummer's identity, extended by Herz (1955), gives

$$\operatorname{etr}(Z)\,{}_1F_1(a;\, b;\, -Z) = {}_1F_1(b-a;\, b;\, Z),$$

where ${}_1F_1(a;\, b;\, Z)$ is a confluent hypergeometric function of matrix argument shown by Constantine (1963) to be expressible as

$$(4.6) \qquad {}_1F_1(a;\, b;\, Z) = \sum_{k=0}^{\infty} \sum_{\kappa\in\mathcal{P}_k} [(a)_\kappa/(b)_\kappa] C_\kappa(Z)/k! .$$

Expanding both sides of (4.6) in zonal polynomials and applying Theorem 1 to the left hand side yields (4.5). □

Let $b = n$ and $a = n - x$. Then we obtain recurrence relations in the $\binom{\lambda}{\varkappa}$ by expanding both sides of the identity

$$(4.7) \qquad \sum_{k=0}^{\ell} (-1)^k \sum_{\varkappa \in \mathcal{P}_k} \binom{\lambda}{\varkappa}(x)_\varkappa/(n)_\varkappa = (n-x)_\lambda/(n)_\lambda$$

in powers of x and n^{-1} and matching coefficients on each side. Now, if $\lambda = (\ell_1, \ldots, \ell_p)$,

$$\begin{aligned}(n-x)_\lambda/(n)_\lambda &= \exp\{\sum_{i=1}^{p} \sum_{j=1}^{\ell_i} \log(1-x/[n+j-\tfrac{1}{2}(i+1)])\} \\ &= \exp\{-\sum_{r=1}^{\infty} (x^r/r) \sum_{j=1}^{p} \sum_{j=1}^{\ell_i} [n+j-\tfrac{1}{2}(i+1)]^{-r}\} \\ &= \exp\{-\sum_{r=1}^{\infty} (x^r/r) \sum_{k=0}^{\infty} [(r)_k(-1)^k/(n^{r+k}k!)]d_{k+1}(\lambda)\},\end{aligned}$$

where $d_{k+1}(\lambda)$ is defined by (2.5). After some simplification, we find

$$\begin{aligned}(4.8) \qquad (n-x)_\lambda/(n)_\lambda = 1 &+ [\sum_{r=1}^{\infty} (-1)^r d_r(\lambda)n^{-r}]x \\ &+ \sum_{s=2}^{\ell} (-1)^s x^s \binom{\ell}{s}n^{-s} - d_2(\lambda)\binom{\ell-1}{s-1}n^{-s-1} \\ &+ [\tfrac{1}{2}(d_2^2(\lambda) + d_3(\lambda))\binom{\ell-2}{s-2} + d_3(\lambda)\binom{\ell-2}{s-1}]n^{-s-2} \\ &- [(\tfrac{1}{6})(d_2^3(\lambda) + 3d_2(\lambda)d_3(\lambda) + 2d_4(\lambda))\binom{\ell-3}{s-3} \\ &+ (d_2(\lambda)d_3(\lambda) + d_4(\lambda))\binom{\ell-3}{s-2} + d_4(\lambda)\binom{\ell-3}{s-1}]n^{-s-3} + O(n^{-s-4}).\end{aligned}$$

Define coefficients $f_r(\varkappa)$ and $Q_s(\varkappa)$ by

$$(4.9) \qquad (x)_\varkappa = \sum_{r=0}^{k} f_{k-r}(\varkappa)x^r, \quad (n)_\varkappa^{-1} = \sum_{s=0}^{\infty} Q_s(\varkappa)n^{-s-k}.$$

Then

$$(4.10) \qquad \sum_{k=0}^{\ell} (-1)^k \sum_{\varkappa \in \mathcal{P}_k} \binom{\lambda}{\varkappa}(x)_\varkappa/(n)_\varkappa = \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} [\sum_{k=s}^{s+t}(-1)^k \sum_{\varkappa \in \mathcal{P}_k} \binom{\lambda}{\varkappa}f_{k-s}(\varkappa)Q_{t-k+s}(\varkappa)] \cdot (x^s/n^{s+t}).$$

Matching coefficients of $x^s n^{-s-t}$ in (4.8) and (4.10), observing that $Q_0(\varkappa) = f_0(\varkappa) = 1$, $Q_j((0)) = Q_j((1)) = 0$, $j \geq 1$, we obtain

$$(4.11) \qquad \sum_{\sigma \in \mathcal{P}_s} f_{s-1}(\sigma)\binom{\lambda}{\sigma} = d_s(\lambda) - \sum_{k=1}^{s-2} (-1)^k \sum_{\rho \in \mathcal{P}_{s-k}} Q_k(\rho) f_{s-k-1}(\rho)\binom{\lambda}{\rho}$$

$$(4.12) \qquad \sum_{\sigma \in \mathcal{P}_s} f_1(\sigma)\binom{\lambda}{\sigma} = \binom{\ell-1}{s-2} d_2(\lambda) + \sum_{\rho \in \mathcal{P}_{s-1}} Q_1(\rho)\binom{\lambda}{\rho}$$

$$(4.13) \qquad \sum_{\sigma \in \mathcal{P}_s} f_2(\sigma)\binom{\lambda}{\sigma} = \binom{\ell-2}{s-3} d_3(\lambda) + \tfrac{1}{2}\binom{\ell-2}{s-4}[d_2^2(\lambda) + d_3(\lambda)]$$
$$+ \sum_{\rho \in \mathcal{P}_{s-1}} Q_1(\rho) f_1(\rho)\binom{\lambda}{\rho} - \sum_{\rho' \in \mathcal{P}_{s-2}} Q_2(\rho')\binom{\lambda}{\rho'}$$

$$(4.14) \qquad \sum_{\sigma \in \mathcal{P}_s} f_3(\sigma)\binom{\lambda}{\sigma} = \binom{\ell-3}{s-4} d_4(\lambda) + \binom{\ell-3}{s-5}[d_2(\lambda) d_3(\lambda) + d_4(\lambda)]$$
$$+ \frac{1}{6}\binom{\ell-3}{s-6}[d_2^3(\lambda) + 3d_3(\lambda) + 2d_4(\lambda)]$$
$$+ \sum_{\rho \in \mathcal{P}_{s-1}} Q_1(\rho) f_2(\rho)\binom{\lambda}{\rho} - \sum_{\rho' \in \mathcal{P}_{s-2}} Q_2(\rho') f_1(\rho')\binom{\lambda}{\rho'}$$
$$+ \sum_{\rho'' \in \mathcal{P}_{s-3}} Q_3(\rho'')\binom{\lambda}{\rho''}.$$

It can be verified that these relations, together with (4.4), suffice to confirm the expressions in (2.12), and could be used to find expressions for $\binom{\lambda}{\varkappa}$ for all $\varkappa \in \mathcal{P}_4$.

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